On generalized inverses and Green's relations

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Abstract

We study generalized inverses on semigroups by means of Green's relations. We first define the notion of inverse along an element and study its properties. Then we show that the classical generalized inverses (group inverse, Drazin inverse and Moore-Penrose inverse) belong to this class.

There exist many specific generalized inverses in the literature, such as the group inverse, the Moore-Penrose inverse [1] or the Drazin inverse ([2], [1]). Necessary and sufficient conditions for the existence of such inverses are known ([4], [2], [5], [6], [7], [8], [14], [9]), as are their properties. If one looks carefully at these results, it appears that these existence criteria all involve Green'relations [4], and that all inverses have double commuting properties. So one may wonder whether we could unify these different notions of invertible.

We propose here to define a new type of generalized inverse, the inverse along an element, that is based on Green's relation's $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ [4], and

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the related notion of trace product ([10], [13]). It appears that this notion encompass the classical generalized inverses but is of richer type. By deriving general existence criteria and properties of this inverse, we will then recover in a common framework the classical results. The framework is the one of semigroups, hence the results are directly applicable in rings or algebras.

This article is divided as follows: in the first section, we review the principal definitions and theorems we will use regarding generalized inverses and Green's relations. In the second section we define our new generalized inverse, the inverse along an element, and derive its properties. In the third section we finally show that the classical generalized inverses belong to this class, and retrieve their properties.

1 Preliminaries

As usual, for a semigroup S, S^1 denotes the monoid generated by S, and $E(S)$ its set of idempotents. We first review the various notions of generalized invertibility and then recall Green's relations, together with some results linked with invertibility.

Generalized inverses

Basically, a generalized inverse is an element that share some (but not all) the properties of the classical inverse in a group. We review here the classical notions.

Let $a \in S$. The element a in S is called regular if $a \in aAa$, that is there exists b such that $aba = b$. In this case b is known as an inner inverse of a. If there exists $b \in S$, $bab = b$ then b is an called an outer inverse of a. An element b that is both an inner and an outer inverse is usually simply called an inverse of a. If it verifies only one of the two conditions, it is called a generalized inverse. The three common generalized inverses are defined by imposing additional properties.

If b is an inverse (inner and outer) of a that commutes with a then b is a called a commuting inverse (or group inverse) of a. Such an inverse is unique and usually denoted by $a^\sharp$. Its name "group inverse" comes from the following result:

Corollary 1.1 ((corollary 4 p. 275 in [10])) *If a and a' are mutually inverse elements of S then $aa' = a'a$ if and only if a and a' belong to the same $\mathcal{H}$-class H. If this be the case, H is a group, and a and a' are inverses therein in the sense of group theory.*

To study non regular elements, Drazin [2] introduced a another a commuting generalized inverse, that is not inner in general. $a \in S$ is Drazin invertible if there exists $b \in S$ and $m \in \mathbb{N}^*$:

1. $ab = ba$;
2. $a^m = a^{m+1}b$;
3. $b = b^2a$.

A Drazin inverse of a is unique if it exists and will be denoted by a^D in the sequel.

Finally, when S is a endowed with an involution $*$ that makes it an involutive semigroup (or $*$-semigroup), *i.e.* the involution verifies $(a^*)^* = a$ and $(ab)^* = b^*a^*$, Moore [11] and Penrose [15] studied inverses b of a with the additional property that $(ab)^* = ab$ and $(ba)^* = ba$. Once again this inverse, if it exists, is unique. It is usually called the Moore-Penrose inverse (or pseudo-inverse) of a and will be denoted by a^+.

Green's relations

For elements a and b of S, Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are defined by:

1. $a\mathcal{L}b$ if and only if $S^1a = S^1b$.

2. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.

3. $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

That is, a and b are $\mathcal{L}$-related (resp. $\mathcal{R}$-related) if they generate the same left (resp. right) principal ideal, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These are equivalence relations on S, and we denote the $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class) of a by $\mathcal{L}_a$ (resp. $\mathcal{R}_a, \mathcal{H}_a$). The $\mathcal{R}$ and $\mathcal{L}$ relations are dual to one another and left (resp. right) compatible.

In addition to Green's relation it is useful to our purpose to introduce their generalization $\mathcal{R}^*$ and $\mathcal{L}^*$: $a\mathcal{R}^*b$ if $a\mathcal{R}b$ in some oversemigroup of S. By lemma 1.1 in [3], this is equivalent to

$$(xa = ya \iff xb = yb \,\forall x, y \in S^1).$$

Relation $\mathcal{L}^*$ is defined dually. We have $\mathcal{R} \subset \mathcal{R}^*$ and $\mathcal{L} \subset \mathcal{L}^*$.

Finally we will also need the notion of trace product: for $a, b \in S$, we say that ab is a trace product if $ab \in \mathcal{R}_a \cap \mathcal{L}_b$.

We will use the following results:

Theorem 1.2 ((Theorem 3 p. 277 [10])) *Let $a, b \in S$. ab is a trace product if and only if $\mathcal{R}_b \cap \mathcal{L}_a$ contains an idempotent element; if this be the case then*

$$aH_b = H_ab = H_aH_b = H_{ab} = \mathcal{R}_a \cap \mathcal{L}_b.$$

Lemma 1.3 ((lemma 4 p. 272 in [10])) *An idempotent element e of S is a right identity element of $\mathcal{L}_e$, a left identity element of $\mathcal{R}_e$ and a two-sided identity element of $\mathcal{H}_e$.*

Corollary 1.4 ((corollary 1 p. 272 and corollary 1 p. 273 in [10]))

1. *No $\mathcal{H}$-class contains more than one idempotent element;*

2. *an $\mathcal{H}$-class $\mathcal{H}_b$ contains an inverse of $a \in S$ if and only if both the $\mathcal{H}$-class $\mathcal{R}_a \cap \mathcal{L}_b$ and $\mathcal{R}_b \cap \mathcal{L}_a$ contain idempotent elements;*

3. *no $\mathcal{H}$-class contains more than one inverse of $a \in S$.*

To be exact, statement 2. is not precisely the same as in [10] where it is assumed that a is regular. But this is a direct consequence of the existence of an idempotent element in $\mathcal{R}_a$ by Von Neumann's lemma 6 [12].

Theorem 1.5 ((Theorem 7 p. 169 in [4]))

1. *If a $\mathcal{H}$-class contains an idempotent e, then it is a group with e as the identity element.*

2. *If for any $a, b \in S$, a, b and ab belong to the same $\mathcal{H}$-class H, then H is a group.*

2 A new generalized inverse: the inverse along an element

Theorem 2.1 *Let $a, d \in S$. Then four following statements are equivalent:*

1. *there exists $b \in S$ verifying $bad = d = dab$ and $b \in dS \cap Sd$;*

2. *there exists $b \in S$ verifying b is outer inverse of a and $b\mathcal{H}d$;*

3. *there exists $b \in S$, there exists an idempotent $e \in \mathcal{R}_d$, such that b is an inner and outer inverse of ae and $b\mathcal{H}d$;*

4. *there exists $b \in S$, there exists an idempotent $f \in \mathcal{L}_d$, such that b is an inner and outer inverse of fa and $b\mathcal{H}d$.*

Moreover, if $b \in S$ verifies one of the four statements, it verifies the four simultaneously.

Proof [1. $\Rightarrow$ 2.] let $b, x, y \in S$ such that $bad = d = dab$ and $b = dx = yd$. Then $bab = badx = dx = b$ and b is an outer inverse of a. but $bad = d = dab$ implies that $d \in bS \cap Sb$ and finally $b\mathcal{H}d$.
[2. $\Rightarrow$ 3.] Let $b \in S$, $bab = b$ and $b\mathcal{H}d$. Then one verifies easily that ba is an idempotent in $\mathcal{R}_d$. First, $bab = b$ implies $baba = ba$ and ba is idempotent. Second, since $b\mathcal{H}d$, there exists $x \in S,\ d = bx$. But $bab = b$ and it follows that $d = bx = babx$. Also there exists $y \in S,\ dy = b$, and $ba = dya$. Finally, b is an inner and outer inverse of $a(ba)$ by direct calculations.
[3. $\Rightarrow$ 4.] Suppose b is the generalized inverse of ae, with $e \in \mathcal{R}_d \cap E(S)$. Since $b \in \mathcal{H}_d,\ \exists x, y, x', y' \in S^1,\ b = dx = x'd,\ d = by = y'b$. Pose $f = xaed$. Then f is idempotent ($ff = xaedxaed = xaebaed = xaed = f$) and $f\mathcal{L}d$ ($f = x'aed,\ d = by = baeby = dxaeby = df$).
Note that by lemma 1.3, $bf = b$ and $eb = b$. It follows that $baeb = bab = bfab = b$. But

$$fabfa = faba = xaedaba = xaey'baeba = xaey'ba = fa$$

and finally b is an inner and outer inverse of fa.
[(4) $\Rightarrow$ (1)] Suppose b is the generalized inverse of fa, with $f \in \mathcal{L}_d \cap E(S)$. First, $b\mathcal{H}d$ implies that $b \in dS \cap Sd$. Also there exists $y, y' \in S,\ d = by = y'b$. But $bab = b$ (f is a right identity on $\mathcal{L}_d = \mathcal{L}_b$ by lemma 1.3 and it follows that $bad = baby = by = d = y'b = y'bab = dab$. This ends the proof.

Definition 2.2 *Let $a, d \in S$. We say that $b \in S$ is an inverse of a **along d** if it verifies one of the four equivalent statements of theorem 2.1. If moreover the inverse b of a along d verifies $aba = a$, we say that b is an **inner** inverse of a along d.*

Note that we have also proved that ba and ab are idempotents in the $\mathcal{R}$ and $\mathcal{L}$-class of d respectively:

Corollary 2.3 *$ba \in \mathcal{R}_d \cap E(S)$ and $ab \in \mathcal{L}_d \cap E(S)$.*

Example 2.4 *Let $S = T_3$ be the full transformation semigroup, that consists of all functions from the set $\{1, 2, 3\}$ to itself with multiplication the composition of functions. We write (abc) for the function which sends 1 to a, 2 to b, and 3 to c.*
The egg-box diagram form T_3 is as follows ($\mathcal{R}$-classes are rows, $\mathcal{L}$-classes columns and $\mathcal{H}$-classes are squares; bold elements are idempotents):

(1 1 1)	***(2 2 2)***	***(3 3 3)***				
			(1 2 2), *(2 1 1)*	***(1 3 3)***, *(3 1 1)*	*(2 3 3)*, *(3 2 2)*	
			(2 1 2), ***(1 2 1)***	*(3 1 3)*, *(1 3 1)*	***(3 2 3)***, *(2 3 2)*	
			(2 2 1), *(1 1 2)*	*(3 3 1)*, ***(1 1 3)***	*(3 3 2)*, ***(2 2 3)***	
						(1 2 3), *(2 3 1)*, *(3 1 2)*, *(1 3 2)*, *(3 2 1)*, *(2 1 3)*

For instance $\mathcal{H}_{(232)} = \{(232), (323)\}$. Direct computations give that $a = (221)$ is (inner) invertible along $d = (232)$ with inverse $b = (323)$, and that $a' = (122)$ and $a'' = (123)$ are invertible along $d = (232)$ but not inner invertible (the inverse is $b = (323)$). (111) is not invertible along (232).

We now state our second theorem, that deals with existence and uniqueness of inverses along an element:

Theorem 2.5 *Let $a, d \in S$.*

1. *a is invertible along d if and only if there exists $e \in (\mathcal{R}_d \cap E(S))$, $(ae)\,d$ and $d\,(ae)$ are trace products.*
2. *If an inverse along d exists, it is unique.*

Proof

EXISTENCE: Suppose a is invertible along d and let $e \in \mathcal{R}_d \cap E(S)$ be the associated idempotent. Then by corollary 1.4 the $\mathcal{H}$-classes $\mathcal{R}_{ae} \cap \mathcal{L}_d$ and $\mathcal{L}_{ae} \cap \mathcal{R}_d$ contain idempotents elements and by theorem 1.2 $(ae)\,d$ and $d\,(ae)$ are trace products.
Suppose now there exists $e \in (\mathcal{R}_d \cap E(S))$, $(ae)\,d$ and $d\,(ae)$ are trace products. Then by corollary 1.4 ae admits a generalized inverse in $\mathcal{H}_d$ and a is invertible along d.

UNIQUENESS: Let b and c be two generalized inverses of a along d. Then $c = (ba)c = b(ac)b$ by lemma 1.3 and lemma 2.3.

The uniqueness of the inverse along an element allows us to introduce the following notations: if it exists, we note $a^{\angle d}$ the inverse of a along d and $a^{\measuredangle d}$ if it is a inner inverse ($aba = a$).

Corollary 2.6 *Let $a, d \in S$. a is inner invertible along d if and only if ad and da are trace products.*

Proof Suppose a is inner invertible along d with inverse b. Pose $e = ba$. Then $e \in \mathcal{R}_d \cap E(S)$ by corollary 2.3, and by theorem 2.5 $(ae)\,d$ and $d\,(ae)$ are trace products. But $ae = aba = a$ since b is an inner inverse of a and ad and da are trace products.
Conversely, if ad and da are trace products, then corollary 1.4 gives the desired result.

Example 2.7 *Let once again $S = T_3$ be the full transformation semigroup of example 2.4. Then the inner invertibility of $a = (221)$ along $d = (232)$ follows from the existence of the idempotent $e = (121)$ in the $\mathcal{H}$-class $\mathcal{R}_d \cap \mathcal{L}_a$ and of the idempotent $f = (223)$ in the $\mathcal{H}$-class $\mathcal{R}_a \cap \mathcal{L}_d$.*

Example 2.8 *Let $\mathcal{S}$ be the subsemigroup of $\mathcal{M}_3(\mathbb{N})$ generated by the matrices*

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $a\mathcal{R}b\mathcal{R}c\mathcal{R}d$ (the semigroup is right simple), $a\mathcal{L}c$ and $b\mathcal{L}d$. Since a and d are idempotents, each $\mathcal{L}$-class contains idempotents elements and it follows that any product of two elements is a trace product. Finally any element is invertible along another one, or equivalently any $\mathcal{H}$-class $\{a, c\}$, $\{b, d\}$ is an inverse transversal (see note 3.16 in [16]).

As a corollary, we get an interesting characterization of the inverse of a along d in terms of the group inverse $(ad)^\sharp$ (or $(da)^\sharp$), as well as a second existence criteria:

Corollary 2.9 *Let $a, d \in S$. The three following statements are equivalent:*

1. *a is invertible along d;*
2. *$ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group;*
3. *$da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group.*

In this case $a^{\angle d} = d(ad)^\sharp = (da)^\sharp d$

Proof

1. $\Rightarrow$ 2. First, suppose a is invertible along d and let e be the idempotent in $\mathcal{R}_d$ such that b is the inverse of ae in $\mathcal{H}_d$. By theorem 2.5, $(ae)d$ is a trace product and it follows that $ae\mathcal{H}_d = \mathcal{H}_{aed}$ or equivalently since e is a right identity on $\mathcal{H}_d$, $a\mathcal{H}_d = \mathcal{H}_{ad}$. Since $b \in \mathcal{H}_d$ and ab is idempotent, the $\mathcal{H}$-class $\mathcal{H}_{ad}$ contains the idempotent ab hence it is a group. Finally equality $bad = d$ gives $ad\mathcal{L}d$.

2. $\Rightarrow$ 3. Suppose now $ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group, and let $x \in S$ such that $d = xad$. From $ad = ad(ad)^\sharp ad = (ad)(ad)(ad)^\sharp = (ad)^\sharp(ad)(ad)$ we get $d = xad = x(ad)(ad)(ad)^\sharp = (da)d(ad)^\sharp$ and $da\mathcal{R}d$. To prove that $\mathcal{H}_{da}$ is a group, by theorem 1.5 we only need to prove that $da\mathcal{H}(da)^2$. But

$$da = xada = x(ad)(ad)(ad)^\sharp a = x(ad)(ad)(ad)^\sharp(ad)(ad)^\sharp a = d(ad)(ad)(ad)^\sharp(ad)^\sharp a$$

and $da\mathcal{R}(da)^2$. But since $d = dad(ad)^\sharp$ we have also

$$da = dad(ad)^\sharp a = d(ad)^\sharp(ad)(ad)(ad)^\sharp a = d(ad)^\sharp(ad)^\sharp(ad)(ad)a$$

and $da\mathcal{L}(da)^2$. Finally $da\mathcal{H}(da)^2$.

3. $\Rightarrow$ 1. Suppose now $da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group. Pose $b = (da)^\sharp d$ and let $x \in S$ such that $d = dax$. Then $bad = (da)^\sharp dad = (da)^\sharp dadax = dax = d$, and $dab = da(da)^\sharp d = da(da)^\sharp dax = dax = d$.
But also $b = (da)^\sharp d = (da)^\sharp dax = da(da)^\sharp x$ and $b \in dS \cap Sd$. Finally $b = (da)^\sharp d$ is the inverse of a along d.

Example 2.10 *Let $\mathcal{S}$ be the subsemigroup of $\mathcal{M}_3(\mathbb{N})$ of example 2.8. The inverse of b along c is $b^{anglec} = c(bc)^\sharp = ca^\sharp = ca = c$.*

We finally prove an interesting result regarding commutativity. If A is a subset of the semigroup S, A' denotes as usual the commutant of A and A'' its bicommutant.

Theorem 2.11 *Let $a, d \in S$ and pose $A = (a, d)$. If a is invertible along d, then $a^{\angle d} \in A''$.*

Proof Let b be the inverse of a along d. It then verifies $bad = d = dab$ and $b \in dS \cap Sd$. Let $x, y \in S$ such that $b = dx = yd$. Suppose $c \in A'$. Then

$$\begin{aligned} cd &= cbad = cdab = dacb \\ &= dc = badc = bcad = dabc \end{aligned}$$

hence $cbad = bcad, \quad dabc = dacb$. Then

$$\begin{aligned} cb &= cbab = cbadx = bcadx = bcab \\ &= bacb = ydacb = ydabc = babc = bc \end{aligned}$$

and $b \in A''$.

Remark 2.12 *If $da = ad$, the two previous results give that $b = a^{\angle d}$ commutes with a and d and that $\mathcal{H}_d = \mathcal{H}_{ad}$ is a group.*

3 Inverses along d and classical inverses

One of main interest of this notion of inverse along an element is that the classical generalized inverses belong to this class:

Theorem 3.1 *Let $a \in S$. (S is a $*$-semigroup in 3.)*

1. *a is group invertible if and only if it is invertible along a. In this case the inverse along a is inner and coincide with the group inverse.*
2. *a admits a Drazin inverse if and only if it is invertible along some a^m, $m \in \mathbb{N}$, and in this case the two coincide.*
3. *a is Moore-Penrose invertible if and only if it is invertible along a^*. In this case the inverse along a^* is inner and coincide with the Moore-Penrose inverse.*

Proof

Group inverse: Suppose a is group invertible. Then $\mathcal{H}_a$ is a group that contains the group inverse $a^\sharp$. It follows that a is invertible along a, with inverse $a^{\measuredangle a} = a^\sharp$.
Conversely, if $a^{\angle a}$ exists, then by corollary 2.9 $\mathcal{H}_a = \mathcal{H}_{a^2}$ is a group. a is then group invertible with inverse $a^\sharp \in H_a$ and by uniqueness of the inverse along a, $a^{\angle a} = a^\sharp = a^{\measuredangle a}$.

Drazin inverse: Suppose a Drazin invertible (with Drazin inverse a^D). Then (Theorem 7 in [2]) there exists $m \in \mathbb{N}^*$, there exists e idempotent in $\mathcal{H}_{a^m}$, $ae = ea \in \mathcal{H}_{a^m}$. Then $\mathcal{H}_{a^m}$ is a group (theorem 1.5) and a is invertible along a^m (take e for idempotent). Moreover (see the proof of Theorem 7 in [2]), the inverse of ae in $\mathcal{H}_{a^m}$ is precisely the Drazin inverse, hence $a^D = a^{\angle a^m}$.
Conversely, suppose there exists $m \in \mathbb{N}^*$, a invertible along a^m. Then by corollary 2.9 $a^{m+1}\mathcal{H}a^m$. But (Theorem 4 p 510 in [2]) a is Drazin

invertible if and only if it is strongly π-regular, *i.e.* there exists $m \in \mathbb{N}^*$, $x, y \in S^1$,

$$a^{m+1}x = a^m = ya^{m+1}$$

or using Green's relation if and only if there exists $m \in \mathbb{N}^*, a^{m+1}\mathcal{H}a^m$. Finally a is Drazin invertible and by the previous result, the two inverses coincide.

<u>M-P inverse:</u> Suppose a Moore-Penrose invertible with Moore-Penrose inverse a^+. Then

$$a^+ = (a^+a)a^+ = \left(a^+a\right)^* a^+ = a^* \left(a^+\right)^* a^+$$

$$a^+ = a^+ \left(aa^+\right) = a^+ \left(aa^+\right)^* = a^+ \left(a^+\right)^* a^*$$

$$a^* = \left(aa^+a\right)^* = \left(a^+a\right)^* a^* = a^+aa^*$$

$$a^* = \left(aa^+a\right)^* = a^* \left(aa^+\right)^* = a^*aa^+$$

These four relations imply that $a^+\mathcal{H}a^*$, hence a is inner invertible along a^* with $a^{\measuredangle a^*} = a^+$.

Conversely, suppose a is invertible along a^* with inverse b. Let $e \in \mathcal{R}_{a^*} \cap E(S)$. Then $(ae)\,a^* = aa^*$ is a trace product and $aa^*\mathcal{L}a^*$, and also by transposition $aa^*\mathcal{R}a$. Finally aa^* is a trace product.

But working with f idempotent in $\mathcal{L}_{a^*}$ give that $a^*\,(fa) = a^*a$ is a trace product and $a^*a\mathcal{R}a^*$, and also by transposition $a^*a\mathcal{L}a$. a^*a is then also a trace product and the inverse is inner.

We finally verify that ab and ba are hermitian using corollary 2.9: $ab = aa^*(aa^*)^\sharp = (aa^*)^\sharp aa^* = b^*a^* = (ab)^*$ and $ba = (a^*a)^\sharp(a^*a) = (a^*a)(a^*a)^\sharp = a^*b^* = (ba)^*$.

Combining theorem 2.5 (or corollary 2.9) and theorem 3.1, we then get directly the following existence criteria and commuting relations for the classical inverses [4], [2], [5], [6], [7], [8], [14], [9]:

Corollary 3.2 *Let $a \in S$.*

1. *A group inverse $a^\sharp$ exists if and only if a^2Ha, and $a^\sharp \in (a)''$.*
2. *A Drazin inverse a^D exists if and only if there exists $m \in \mathbb{N}^*, a^{m+1}\mathcal{H}a^m$, and $a^D \in (a)''$.*

3. *A Moore-Penrose inverse a^+ exists if and only if $aa^* \in \mathcal{R}_a$ and $a^*a \in \mathcal{L}_a$, and $a^+ \in (a, a^*)''$.*

Note that many other results involving classical inverses are then straightforward consequences of theorem 3.1. We give two instances of this:

Proposition 3.3 ((proposition 2 p. 162 in [14])) *Given a in a ring R with involution $*$, the following conditions hold:*

1. *If $aR = a^*R$ then a^+ exists with respect to $*$ iff $a^\sharp$ exists, in which case $a^+ = a^\sharp$.*
2. *If a^+ exists with respect to $*$, $a^\sharp$ exists and $a^+ = a^\sharp$ then $aR = a^*R$.*

Proof By transposition, $(aR = a^*R) \iff (Ra^* = Ra) \iff (a\mathcal{H}a^*)$. Since the inverse along an element d depends only on the $\mathcal{H}$-class of d, theorem 3.1 then give the desired result.

Remark that in rings with involution, $(aR = a^*R) \iff (aa^* = a^*a)$, and we could have used corollary 2.9 instead.

Theorem 3.4 ((Theorem 5.3 p. 144 in [8])) *Let R be a ring with involution $*$. An element $a \in R$ is Moore-Penrose invertible if and only if $aa^*\mathcal{R}^*a$, $a^*a\mathcal{L}^*a$ and a^*a is group invertible. Then also aa^* is group invertible and*

$$a^+ = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp$$

Proof Suppose a^+ exists. Then a is invertible along a^*. By corollary 2.9, $\mathcal{H}_{aa^*}$ and $\mathcal{H}_{a^*a}$ are groups and

$$a^{\angle d} = (a^*a)^\sharp a^* = a^*(aa^*)^\sharp$$

Also by corollary 2.9 $aa^* \in \mathcal{R}_a$ and $a^*a \in \mathcal{L}_a$. $\mathcal{R} \subset \mathcal{R}^*$ and $\mathcal{L} \subset \mathcal{L}^*$, hence $aa^*\mathcal{R}^*a$, $a^*a\mathcal{L}^*a$.
Conversely, suppose $aa^*\mathcal{R}^*a$, $a^*a\mathcal{L}^*a$ and a^*a is group invertible. Then $\mathcal{H}_{a^*a}$ is a group hence it contains a idempotent $f = xa^*a$ that verifies $f\mathcal{L}^*a$. . From $ff = f$ we get $af = a$, and finally $a\mathcal{L}f\mathcal{L}a^*a$. The conclusion then follows from corollary 2.9 and theorem 3.1.

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